Key phenomena:

1. changing Φ_m magnetic flux loop causes induced EMF around loop

$$\mathcal{E}_{ind} = -\frac{d\Phi_m}{dt}$$

if loop is a conductor w/ resistance R, an induced current will flow

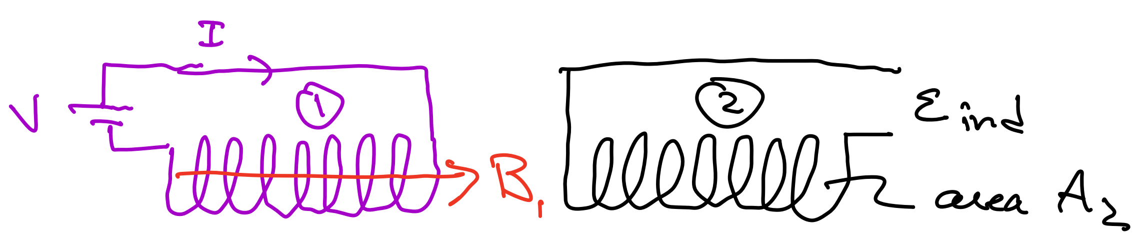
$$I_{ind} = \frac{\mathcal{E}_{ind}}{R}$$

2. external currents in wires cause magnetic fields

solenoid: $B = \mu_0 n I$ along center

wire: $B = \frac{\mu_0 I}{2\pi r}$ r = dist from wire

next put them together with external currents causing induced currents



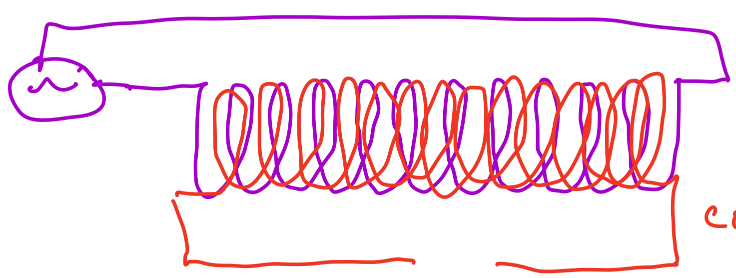
- current thru coil 1 causes B field
- B_1 will go thru coil 2, causing flux Φ_2

$$\Phi_2 = B_1 N_2 A_2 \qquad N_2 = \#\text{ turns in coil 2}$$

- if B doesn't change, then $\frac{d}{dt}\Phi_2 = 0$ so no induced $\mathcal{E}_2$ in coil 2
- if you move coil 1 closer to coil 2 you will cause a $\frac{d\Phi_2}{dt} \Rightarrow \mathcal{E}_2$ induced in 2

$\Rightarrow$ these coils are said to be "flux linked"

now vary I_1 using variable voltage source, and couple the coils so it's easy to calculate the flux



say they have same length,

so $n_1 = \frac{N_1}{L}$ & $n_2 = \frac{N_2}{L}$

B_1 produced inside coil ①

$$B_1 = \mu_0 n I_1(t)$$

ϕ_2 is flux through 1 turn of coil 2

total flux in coil 2 is $\Phi_2 = N_2 \phi_2$ N_2 loops

$$\phi_2 = B_1 \cdot A_2 = B_1 A \quad (A_1 = A_2)$$

so $\Phi_2 = N_2 B_1 A = N_2 A \mu_0 n_1 I_1(t)$

note: Φ_2 total flux in coil ② due to B_1 from coil ① is proportional to only physical characteristics of coils

$$\Phi_2 = (N_2 n_1 A \mu_0) I_1$$

coils that are flux linked always have this:

$$\Phi_2 \propto I_1$$

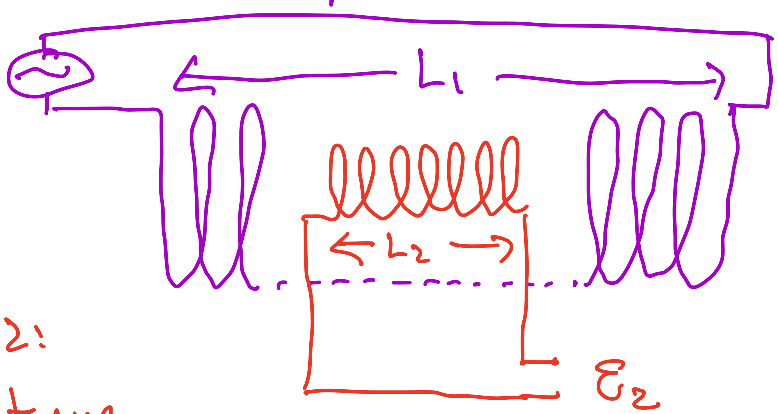
in general it may be difficult to calculate proportionality, but we can usually measure it!

$\Phi_2 = M I_1$ $M \equiv$ "mutual" inductance units of M: $M = \Phi_2 / I_1$ so units are Webers/ampere

this is a Henry

$1\,H = 1\,Wb/A$

ex: 2 coils w/ different areas



coil 1:

N_1 turns

L_1 length

A_1 area

coil 2:

N_2 turns

A_2 area $(A_2 < A_1)$

L_2 length $(L_2 < L_1$

current I_1 generates B_1 constant inside coil 1

$$B_1 = \mu_0 n_1 I_1 \qquad n_1 = N_1 / L_1$$

this B-field goes thru coil 2

$$\phi_2 = B_1 A_2 \quad \text{so} \quad \Phi_2 = N_2 B_1 A_2$$

$$= \underbrace{\mu_0 n_1 N_2 A_2}_{M} \cdot I_1$$

$$M = \Phi_2 / I_1 = \mu_0 n_1 n_2 L_2 A_2 \qquad (n_2 L_2 = N_2)$$

what if instead you put current in coil ② and calculate M?

$$B_2 = \mu_0 n_2 I_2$$

$\phi_1 = B_2 \cdot A_2$! why not $B_2 A_1$? because B_2 only exists inside area A_2, <u>not</u> A_1 which is $> A_2$

so $\Phi_1 = N_1' \phi_1$ where N_1' is the # loops in coil ① that overlap with coil 2

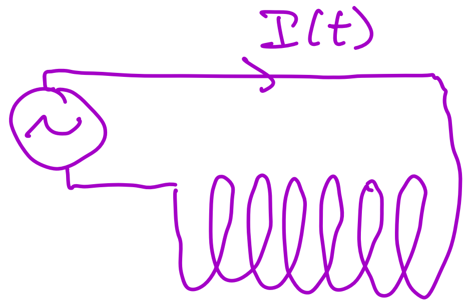
$N_1' = n_1 L_2$ since only L_2 length has overlap

so $\Phi_1 = n_1 L_2 \phi_1 = n_1 L_2 A_2 B_2$

$$= n_1 L_2 A_2 \mu_0 n_2 I_2$$
$$= (\mu_0 n_1 n_2 L_2 A_2) I_2$$
$$= M I_2$$

this is the same as before & is why we call M "mutual inductance"

Self-inductance



single coil w/ variable voltage source generates time varying current

$I(t) \to B(t)$ but since its a function of time, $B(t)$ will change

$B(t) \propto I(t)$

this will produce a time vary flux $\phi(t)$ thru each loop

total flux $\Phi = N\phi$ $N = \#$ turns

self inductance L defined just like M

$$N\phi = LI$$

for solenoid $B = \mu_0 n I$

$$\phi = BA$$

$$\Phi = NBA = \underbrace{\mu_0 N n A}_{L} I$$

since $I(t)$ changes then $\frac{d\Phi}{dt} \neq 0$

so there's a "back" EMF around the loops

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt} LI = -L\frac{dI}{dt}$$

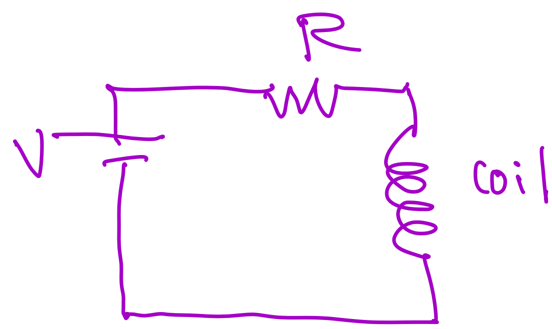
so use $\Phi = LI$ to calculate L

$\mathcal{E} = -\frac{d\Phi}{dt}$ to calculate EMF

coils all have a self inductance L (aka "inductance")

Inductors in circuits

constant voltage source:



neglect resistance of coil ($R_{coil} \lll R$)

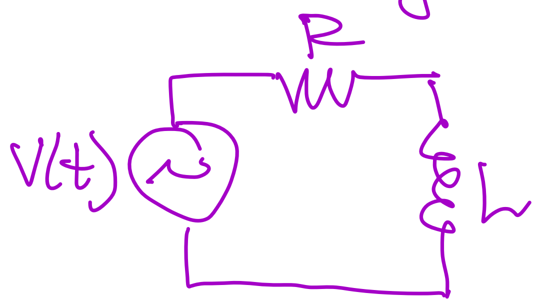
coil is called "an inductor" w/ self inductance L

if V = constant then $I = V/R$ = constant

$\frac{dI}{dt} = 0 \therefore B$ = constant inside L

so no induced EMF around inductor coils

Now make voltage source variable



$\frac{dV}{dt} \neq 0$ so $\frac{dI}{dt} \neq 0$ so $\frac{dB}{dt} \neq 0$

so $\frac{d\Phi}{dt}_{self} \neq 0$ $\therefore$ will be induced $\mathcal{E}$ around inductor coils

$\Rightarrow$ induced current will oppose current driven by voltage source

so net current will be less than I with no inductor $\Rightarrow$ L reduces current just like a resistor with some impedance

$\Rightarrow$ all due to Faraday's law

just like a resistor, must be a voltage drop along direction of current

since I & I_{ind} are opposed, $V_L = L\frac{dI}{dt}$

this preserves energy conservation around the loop

note that as frequency of V(t) increases, $\frac{dI}{dt}$ increases & V_L increases which means induced current increases so <u>net</u> current <u>decreases</u>

→ inductors act like frequency dependent resistor!

Energy picture

in the LR circuit the inductor acts like a resistor in reducing the current.

$\Rightarrow$ this takes energy! inductor is pushing against power supply

power thru any component is always $P = I \cdot V$

where V = voltage drop across it

for inductor $V \equiv \mathcal{E} = L \frac{dI}{dt}$

so $P = LI \frac{dI}{dt} = L \frac{d}{dt} \frac{1}{2} I^2 = \frac{d}{dt}\left(\frac{1}{2} L I^2\right)$

power is always $\frac{d}{dt}$ energy

so the energy stored in inductor coil is

$U_L = \frac{1}{2} L I^2$

for solenoid $L = \mu_0 n N A$

$U = \frac{1}{2} \mu_0 n N A I^2$ here $B = \mu_0 n I$

(use $N = nL$) $= \frac{1}{2} N A \mu_0 n I^2 = \frac{1}{2} n L A \mu_0 n I^2$

$$= LA \cdot \frac{\mu_0 n^2 I^2}{2} = LA \cdot \frac{\mu_0^2 n^2 I^2}{2\mu_0}$$

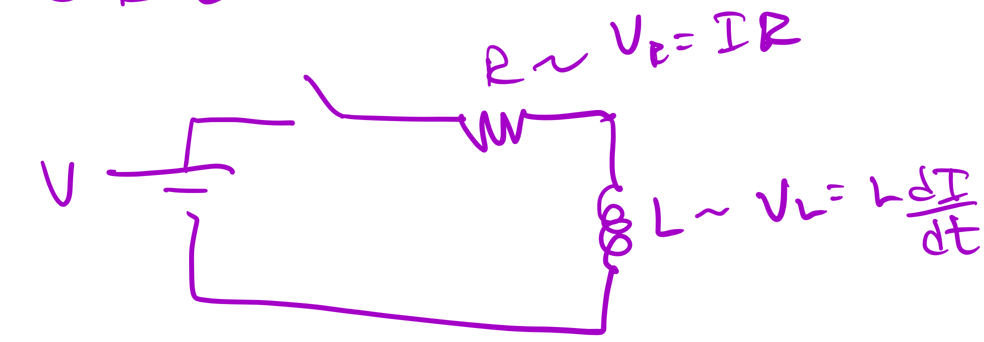
$$= LA \frac{B^2}{2\mu_0}$$ note that LA = volume of solenoid

so $U = \frac{B^2}{2\mu_0} \cdot \text{Volume}$

so $\frac{B^2}{2\mu_0}$ = magnetic energy density

$\Rightarrow$ inductor is storing energy!

DC R-L circuit



close switch: $V = IR + L\frac{dI}{dt}$

solution: diff eqn

$$L\frac{dI}{dt} = V - IR$$

$$\frac{dI}{dt} = \frac{V}{L} - \frac{IR}{L}$$

$\frac{IR}{L}$ must have units of $\frac{A}{s}$

$\therefore$ L/R has units of time

let $\tau = L/R$

write as
$$\frac{dI}{dt} = \frac{V}{R}\frac{R}{L} - \frac{IR}{L}$$
$$= \left(\frac{V}{R} - I\right)\frac{R}{L}$$

so
$$\frac{dI}{\frac{V}{R} - I} = \frac{dt}{\tau}$$

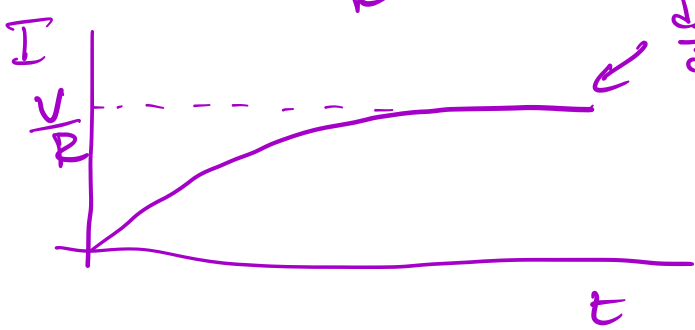
$$\frac{dI}{I - V/R} = -dt/\tau$$

so $\ln(I - V/R) = -\frac{t}{\tau} + k$ indefinite integral

$$I - V/R = e^{k} e^{-t/\tau}$$

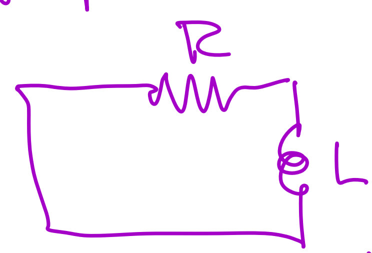
at $t=0$, $I=0$ ∴ $e^{k} = -V/R$

∴ $I = \frac{V}{R}\left(1 - e^{-t/\tau}\right)$



$\frac{dI}{dt} \to 0$ ∴ only voltage drop is across R ∴ $I = V/R$

now jumper V out of circuit leaving only R & L



current will collapse causing $\frac{dI}{dt}$

$V_R + V_L = 0$ now

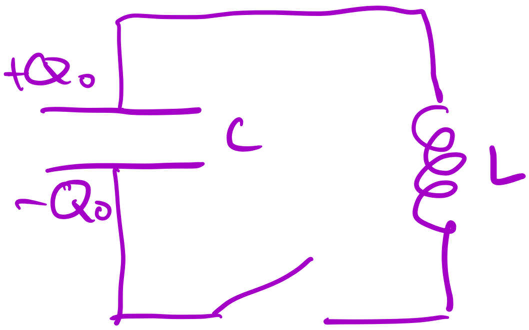
so $IR + L\frac{dI}{dt} = 0 \Rightarrow \frac{I}{\underbrace{L/R}_{\tau}} + \frac{dI}{dt} = 0$

$\frac{dI}{dt} = -\frac{I}{\tau} \Rightarrow I = I_0 e^{-t/\tau}$ where $I_0 = I(t=0)$

current decays exponentially

LC circuit

charge up capacitor by charge +Q on top, −Q on bottom



$C = Q/V$

so $V_C = Q_0/C$

so will be a voltage $V_C = Q_0/C$

- close switch. this will draw current thru inductor causing a voltage drop across C as the charge is reduced.
- current thru L will cause EMF that will generate induced current <u>back</u> towards capacitor
- induced EMF $\mathcal{E} = -L\frac{dI}{dt}$ will be equal to voltage across capacitor as it discharges

$$\frac{Q(t)}{C} = -L\frac{dI(t)}{dt}$$

- after cap discharges (or as it tries to!) then induced current flow L will try to recharge what's happening: energy density inside C (u_E) decreases, increasing u_B inside inductor

But they are out of phase

because $\mathcal{E}_L \propto$ rate of change of I

and $V_c \sim Q$ so I is rate of change of Q

so $V_L \propto$ rate of change of rate of change of V_c !

solve: $$\frac{Q}{C} = -L\frac{dI}{dt} = -L\frac{d^2Q}{dt^2}$$

$$\frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

this is SHM: $\frac{d^2x}{dt^2} + \omega^2 x = 0 \Rightarrow x = x_0 \cos \omega t$

for LC, $\omega^2 = 1/LC$

so $Q = Q_0 \cos \omega t$ $\quad$ ($t=0$, $Q = Q_0 = CV_0$ across C)

and voltage across

charge on capacitor will oscillate

at $t=0$, $Q = Q_0 = CV_0$

so $Q = CV_0 \cos \omega t$

current thru inductor:

$$I = \frac{dQ}{dt} = -CV_0 \omega \sin \omega t$$

in cap: E across $\propto Q$

in ind: $B \propto I$

$\therefore$ E & B "exchange" energy

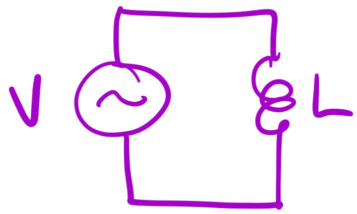
ex: 100 μF capacitor charged up
25 mH inductor

$$LC = 100 \times 10^{-6} \cdot 25 \times 10^{-3} = 2500 \times 10^{-9}$$
$$= 2.5 \times 10^{-12}$$

$$\omega = \frac{1}{\sqrt{LC}} = 632 \times 10^{3} \text{ rad/sec}$$

$f = \omega / 2\pi \simeq 100\,\text{kHz}$ oscillator!

ex:



$V(t) = V_0 \cos\omega t$

conservation of energy: $V = \mathcal{E}_L = L\frac{dI}{dt}$

so $V_0 \cos\omega t = L\frac{dI}{dt}$

this says $I \propto \sin\omega t$

solve: $I = A\sin\omega t + B$

$\frac{dI}{dt} = A\omega\cos\omega t = \frac{V_0\cos\omega t}{L}$

$\therefore A = \frac{V_0}{\omega L}$

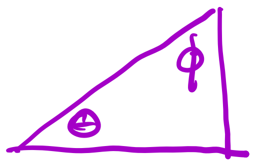
$I(t) = \frac{V_0}{\omega L}\sin\omega t$

① $\frac{V_0}{\omega L}$ ⇒ looks like a current $\therefore R_L = \omega L$ looks like a resistance

for inductor, as $f\nearrow$, $R_L\nearrow$ so inductors do not like high frequencies

$L \rightarrow$ low pass filter

② current and voltage are now out of phase by 90°



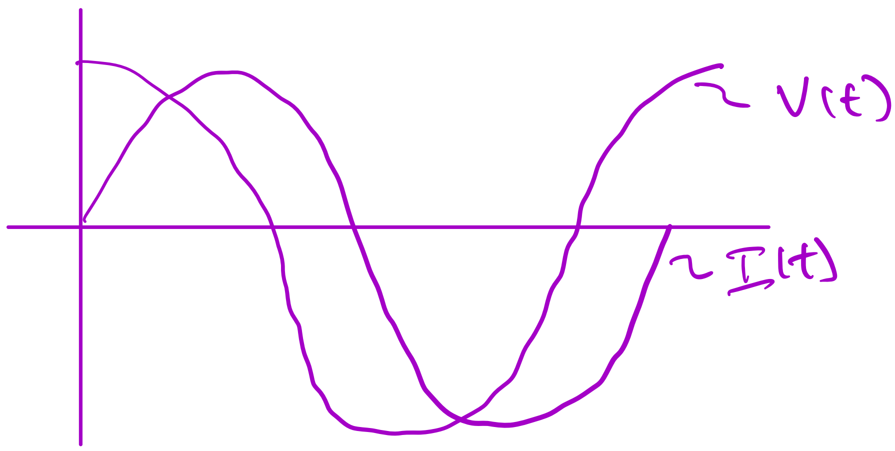
$\cos\theta = \sin\phi$ and $\phi + \theta = 90$

so $\sin\theta = \cos(90 - \theta)$

$= \cos(\theta - 90)$

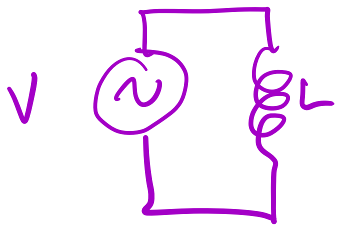
so can write $I(t) = \frac{V_0}{\omega L} \sin \omega t$

$= \frac{V_0}{\omega L} \cos(\omega t - \pi/2)$



I "lags" V by $\pi/2$

AC circuit w/ complex numbers



$V(t) = V_0 \cos \omega t$

make $V(t) = V_0 e^{i\omega t}$

$= V_0 \cos \omega t + V_0 i \sin \omega t$

then keep only real part

$Re(V(t)) = V_0 \cos \omega t$

now solve circuit

$$\underset{\text{gain}}{V} = \underset{\text{loss}}{V_L} = L \frac{dI}{dt}$$

so $V_0 e^{i\omega t} = L \frac{dI}{dt}$

$\therefore \frac{dI}{dt} = \frac{V_0}{L} e^{i\omega t}$

$\therefore I = \frac{V_0}{iL\omega} e^{i\omega t} = -\frac{iV_0}{\omega L} e^{i\omega t}$

$I = -\frac{V_0}{\omega L} (i \cos\theta + i^2 \sin \omega t)$

$= \frac{V_0}{\omega L} (\sin \omega t - i \cos \omega t)$

keep $Re(I) \Rightarrow I = \frac{V_0}{\omega L} \sin \omega t$ as above!

(i) notice: $I = \frac{V_0}{i\omega L} e^{i\omega t} \Rightarrow$ rewrite

$$V_0 e^{i\omega t} = I(i\omega L)$$

or $V = I \underbrace{(i\omega L)}_{\text{"reactance" } X_L}$

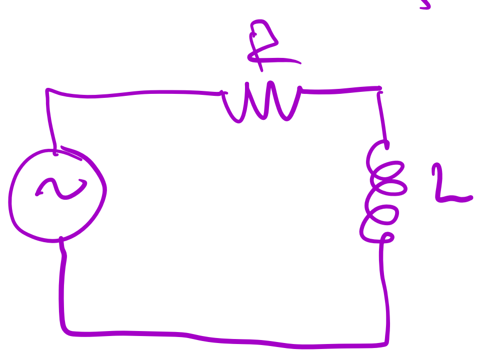
$X_L = i\omega L$ complex impedance

inductors act like resistors in AC circuits w/ complex impedance

so if we replace inductors w/ X_L then we can use "Ohm's Law" for AC circuits easy!

$$V = I X_L$$

L R circuit using complex numbers



$X_R = R$
$X_L = i\omega L$

then $V = IR + IX_L$
$= I(R + i\omega L)$

$$\therefore \; I(t) = \frac{V(t)}{R + i\omega L}$$

now take real part

$$\text{write} \quad \frac{1}{R+i\omega L} = \frac{1}{R+i\omega L} \cdot \frac{R-i\omega L}{R-i\omega L} = \frac{R-i\omega L}{R^2+\omega^2 L^2}$$

$$= \underbrace{\frac{R-i\omega L}{\sqrt{R^2+\omega^2L^2}}} \quad \frac{1}{\sqrt{\quad}}$$

$$\text{let} \quad \cos\phi = \frac{R}{\sqrt{\quad}}, \quad \sin\phi = \frac{\omega L}{\sqrt{\quad}}$$

$$\tan\phi = \omega L / R \qquad \text{let} \quad \tau \equiv L/R$$

$$\tan\phi = \omega\tau$$

$$\text{so} \quad \frac{1}{R+i\omega L} = \frac{\cos\phi - i\sin\phi}{\sqrt{R^2+\omega^2L^2}} = \frac{e^{-i\phi}}{\sqrt{\quad}}$$

$$\therefore \; I(t) = V_0 e^{i\omega t} \cdot \frac{e^{-i\phi}}{\sqrt{\quad}}$$

$$= \frac{V_0}{R} \frac{e^{i(\omega t - \phi)}}{\sqrt{1+\omega^2\tau^2}}$$

$$\text{keep} \quad Re(I) = \frac{V_0}{R} \frac{\cos(\omega t-\phi)}{\sqrt{1+\omega^2\tau^2}}$$

ϕ = phase shift $\Rightarrow$ gets smaller (I in phase w/ V) as $L \to 0$ or $R \to \infty$

① inductor causes a phase change in current wrt driving voltage

$\tan\phi = \frac{\omega L}{R} = \omega\tau$

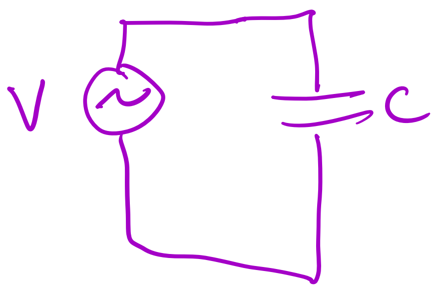
② as $\omega \to 0$, becomes constant voltage

$\omega\tau \to 0$, $\tan\phi \to 0 \Rightarrow \theta \to 0$

$I(t) = \frac{V_0}{R}\cos\omega t$ ✓

③ as $\omega \to$ large, $I \to 0$ because $X_L \to \infty$
acts like a large resistance
so inductor is a low pass filter

AC - capacitor



$$V(t) = V_0 e^{i\omega t}$$

$$V_c = \frac{Q}{C} = V \text{ supply}$$

so $Q = CV$

$$I = \frac{dQ}{dt} = C\frac{dV}{dt} = C\frac{d}{dt} V_0 e^{i\omega t}$$

$$= i\omega C V_0 e^{i\omega t}$$

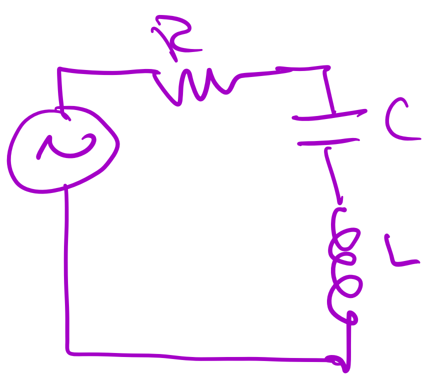
$$= \frac{V_0}{X_C} e^{i\omega t}$$

$\therefore$ $X_C = \frac{1}{i\omega C}$ complex reactance for capacitor

and $X_L = i\omega L$

$X_R = R$

LCR circuit



C & L have "complex" impedances

R has "real" impedance

gain drops

$$V = IR + IX_c + IX_L$$

$$= I\left(R + \frac{1}{i\omega C} + i\omega L\right)$$

$$\therefore I = \frac{V_0 e^{i\omega t}}{R + \frac{1}{i\omega C} + i\omega L}$$

$$R + \frac{1}{i\omega C} + i\omega L = R - \frac{i}{\omega C} + i\omega L = R + i\left(\omega L - \frac{1}{\omega C}\right)$$

$$= R + i\frac{L}{\omega}\left(\omega^2 - \frac{1}{LC}\right)$$